# BOUNDARY ELEMENT SOLUTION OF 2D COUPLED PROBLEM IN ANISOTROPIC PIEZOELECTRIC FGM PLATES

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**Abstract.** The mechanics of the piezoelectric functionally graded material (FGM) has received considerable research effort with their increasing usage in various applications including sensors and actuators, piezoelectric motors, reduction of vibrations and noise, infertility treatment and photovoltaics. It is hard to find the analytical solution of a problem in a general case, therefore, an important number of engineering and mathematical papers devoted to the numerical solution have studied the overall behavior of such materials. The time-stepping dual reciprocity boundary element method was proposed to solve the 2D coupled problem in anisotropic piezoelectric FGM plates. The accuracy of the proposed method was examined and confirmed by comparing the obtained results with those known previously.

## 1 INTRODUCTION

Numerical modelling of piezoelectric solids present certain difficulties since they exhibit not only electro-elastic coupling but anisotropic behaviour. Piezoelectric effect can only appear in crystals that lack of a centre of symmetry and that, as a consequence, are anisotropic. This anisotropy reduces in most cases to transversal isotropy. Piezoelectric ceramics are used for construction of sensors, transducers, actuators as well as adaptative structures. Lead zirconate titanate (PZT) is the most widely used piezoceramic. There are also piezopolymers as the polyvinilidene fluoride (PVDF). Owing to the coupling effects between mechanical and electric properties, piezoelectric materials (PMs) have found wide technological applications as sensors and actuators, piezoelectric motors, reduction of vibrations and noise, infertility treatment and photovoltaics. Applications of piezoelectric materials as electro-mechanical devices have also stimulated a wide range of analytical researches [1-14]. It is usually difficult to obtain analytical solutions to problems involving finite solids or complex boundary conditions. Numerical methods, such as the finite difference method [15-19], the finite element method [20] and the boundary element method (BEM) [21-31] have also been applied to the analysis of electromechanical coupling under complicated conditions. Pan [32] derived the Green's functions for the anisotropic piezoelectric solids in an infinite plane, a half plane, and two joined dissimilar half-planes using the complex variable function method and presented a single-domain BEM analysis of 2D facture mechanics. Recent developments of 2D Green's functions and BEM analysis can be found in Refs. [33–37] for example. It is well known that the BEM presents significant advantages over other numerical techniques for the analysis of fracture mechanics problems. This fact has led to the publication of several BE approaches for the analysis of cracks in piezoelectric solids in the last few years. Presence of domain integrals in the formulation of the BEM dramatically decreases the efficiency of this technique. One of the most frequently used techniques for converting the domain integral into a boundary one developed through our paper is the so-called dual reciprocity boundary element method (DRBEM) [38-46].

In this article the DRBEM is used to solve the coupled problem in anisotropic piezoelectric FGM plates. In the case of two-dimensional, a numerical scheme for the implementation of the method is presented. The accuracy of the proposed method was examined and confirmed by comparing the obtained results with those known before.

## 2 FORMULATION OF THE PROBLEM

Here, we present the basic equations of the piezoelectric elasticity theory, which will be used for the solution of the problem described in the Introduction. With reference to a Cartesian coordinate system (x, y, z) as shown in Fig. 1. We shall consider a functionally graded anisotropic piezoelectric plate. The plate occupies the region $R = \{(x, y, z): 0 < x < \zeta, 0 < y < \Psi, 0 < z < \xi$ with graded material properties in the thickness direction.

In this paper, the FGM properties are graded along the thickness direction ($x$-direction) of the plate. The governing equations for the stress wave propagation in anisotropic functionally graded piezoelectric plate may be written in the following form

$$C_{fghi} u_{h,if} = \rho \ddot{u}_g - C_{fghi} \aleph u_{h,i} - e_{ifg} \left[\Phi_{,if} + \aleph \Phi_{,i}\right] \tag{1}$$

$$e_{fhi} u_{h,if} = \epsilon_{fi} \Phi_{,if} - e_{fhi} \aleph u_{h,i} + \epsilon_{fi} \left[\Phi_{,if} + \aleph \Phi_{,i}\right] \tag{2}$$

where $\sigma_{fg}$ is the mechanical stress tensor, $\varepsilon_{hi}$ is the strain tensor, $u_h$ is the displacement vector, $E_i$ is the electric field vector, $\Phi$ is the electric potential, $D$ is the electric displacement, $C_{fghi}$ is the elasticity tensor $\left(C_{fghi} = C_{gfhi} = C_{hifg}\right)$, $e_{ifg}$ is the piezoelectric tensor $\left(e_{ifg} = e_{igf}, \epsilon_{fi}\right.$ is the permittivity tensor $\epsilon_{fi} = \epsilon_{if}$, $\rho$ is the density and $\aleph = mx + 1$.

A superposed dot denotes differentiation with respect to the time and a comma followed by a subscript denotes partial differentiation with respect to the corresponding coordinates.

## 3 NUMERICAL IMPLEMENTATION

Using the contracted notation, the governing equations (1) and (2) can be combined to form a single equation as follows:

$$L_{\mathrm{GH}} U_{\mathrm{H}} = \rho \delta_{GH} \ddot{U}_{\mathrm{H}} - \mathrm{B_G} \tag{3}$$

where

$$U_{\mathrm{H}} = \begin{cases} u_h & h = \mathrm{H} = 1,2,3 \\ \Phi & \mathrm{H} = 4 \end{cases} \tag{4}$$

$$\mathrm{Z_G} = \begin{cases} t_g & g = G = 1,2,3 \\ q & G = 4 \end{cases} \tag{5}$$

$$C_{f\mathrm{GH}i} = \begin{cases} C_{fghi}, & g = \mathrm{G} = 1,2,3; h = \mathrm{H} = 1,2,3 \\ \mathrm{e}_{ifg}, & g = \mathrm{G} = 1,2,3; \mathrm{H} = 4 \\ \mathrm{e}_{fhi}, & \mathrm{G} = 4; \quad h = \mathrm{H} = 1,2,3 \\ -\epsilon_{fi}, & \mathrm{G} = 4; \quad \mathrm{H} = 4 \end{cases} \tag{6}$$

$$\delta_{GH} = \begin{cases} \delta_{gh} & g = G = 1,2,3, \quad k = K = 1,2,3 \\ 0 & otherwise \end{cases} \tag{7}$$

$$L_{\mathrm{GH}} = C_{f\mathrm{GH}i} \frac{\partial}{x_i} \frac{\partial}{x_f} \tag{8}$$

$$\mathrm{B_G} = \begin{cases} C_{fghi} \bar{\bar{u}} + \mathrm{e}_{ifg} \bar{\bar{\Phi}}, & g = G = 1,2,3 \\ \epsilon_{fi} \Phi_{,if} - \mathrm{e}_{fhi} \bar{\bar{u}} + \epsilon_{fi} \bar{\bar{\Phi}}, & G = 4 \end{cases} \tag{9}$$

$$\bar{\bar{u}} = \aleph u_{h,\mathrm{i}}, \qquad \bar{\bar{\Phi}} = \Phi_{,if} + \aleph \Phi_{,i} \tag{10}$$

Now, we choose the fundamental solution $U_{MH}^{*}$ as weighting function as follows

$$L_{GH} U_{MH}^{*}(x,\xi) = -\delta_{GM} \delta(x,\xi) \tag{11}$$

By integrating the weighted residual formula by parts twice we obtain the following piezoelectric reciprocity relation

$$\int_R \left( L_{\mathrm{GH}} \mathrm{U}_H U_{MG}^{*} - L_{\mathrm{GH}} U_{MH}^{*} \mathrm{U_G} \right) dR = \int_\Gamma \left( U_{MG}^{*} \mathrm{Z_G} - \mathrm{Z}_{MG}^{*} \mathrm{U_G} \right) d\Gamma \tag{12}$$

where

$$\mathrm{Z}_{MG}^{*} = C_{f\mathrm{G}Hi} U_{MH,i}^{*} n_f$$

By the use of sifting property, we obtain from equation (12) the piezoelectric integral representation formula

$$U_{\mathrm{M}}(\xi) = \int_\Gamma \left( U_{MG}^{*} \mathrm{Z_G} - \mathrm{Z}_{MG}^{*} \mathrm{U_G} \right) d\Gamma - \int_R U_{MG}^{*} \left( \rho \delta_{GH} \ddot{U}_{\mathrm{H}} - \mathrm{B_G} \right) dR \tag{13}$$

To transform the domain integrals into boundary integrals over the global boundary of the analyzed domain, the DRBEM can be applied to equation (13) to give the dual reciprocity representation formula of piezoelectric as

$$U_H(\xi) = \int_\Gamma \left( U_{HG}^{*} \mathrm{Z_G} - \mathrm{Z}_{HG}^{*} \mathrm{U_G} \right) d\Gamma + \sum_{q=1}^{N} \left( U_{HN}^{q}(\xi) + \int_\Gamma \left( T_{HG}^{*} U_{GN}^{q} - \mathrm{U}_{HG}^{*} T_{GN}^{q} \right) d\Gamma \right) \alpha_{\mathrm{N}}^{\mathrm{q}} \tag{14}$$

According to the steps described in Fahmy [43], the dual reciprocity boundary integral equation (14) can be written in the following system of equations

$$\bar{\zeta}\breve{U}(t) - \bar{\eta}\breve{T}(t) = \left(\bar{\zeta}\breve{\mho}(t) - \bar{\eta}\breve{\tau}(t)\right)\alpha(t) \tag{15}$$

where $\bar{\zeta}$, $\bar{\eta}$ are BEM system matrices, $\breve{U}$, $\breve{T}$ contain the nodal values of the generalized displacements and fluxes, and $\breve{\mho}$, $\breve{\tau}$ contain the particular solutions

The coefficient vector $\alpha_s(t)$ can be calculated by setting up a system of N equations from (15) using the point collocation procedure, which yields the system

$$M\ddot{\breve{U}}(t) + \bar{\zeta}\breve{U}(t) = \bar{\eta}\breve{T}(t) + \breve{\mathcal{B}}(t) \tag{16}$$

where the volume matrix V, piezoelectric mass matrix $M$ and source vector $\breve{\mathcal{B}}(t)$ are as follows:

$$V = \left(\bar{\eta}\breve{\tau}(t) - \bar{\zeta}\breve{\mho}(t)\right)\mathcal{F}^{-1}, \qquad M = \rho V, \quad \breve{\mathcal{B}}(t) = V\breve{B}(t). \tag{17}$$

The following matrix equation is obtained from Eq. (16).

$$\begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}\begin{bmatrix} \ddot{U}^{k} \\ \ddot{U}^{u} \end{bmatrix} + \begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix}\begin{bmatrix} U^{k}(t) \\ U^{u}(t) \end{bmatrix} = \begin{bmatrix} \bar{\eta}^{11} & \bar{\eta}^{12} \\ \bar{\eta}^{21} & \bar{\eta}^{22} \end{bmatrix}\begin{bmatrix} \mathrm{T}^{k}(t) \\ \mathrm{T}^{u}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{1}(t) \\ \mathcal{B}^{2}(t) \end{bmatrix} \tag{18}$$

The unknown fluxes $\mathrm{T}^{u}(t)$ are obtained from the first row of matrix equation (18) and are expressed as follows.

$$\begin{aligned}\mathrm{T}^{u}(t) = (\eta^{12})^{-1}\big[M^{11}\ddot{U}^{k}(t) + M^{12}\ddot{U}^{u}(t) + K^{11}U^{k}(t) \\ +K^{11}U^{k}(t) + K^{12}U^{u}(t) - \bar{\eta}^{11}\mathrm{T}^{k}(t) - \mathcal{B}^{1}(t)\big]\end{aligned} \tag{19}$$

Making use of Eq. (19), we can write the second row of matrix equation (18) as

$$M^{u}\ddot{U}^{u}(t) + K^{u}U^{u}(t) = Q^{k}(t) \tag{20}$$

where

$$\begin{aligned}
Q^{k}(t) &= \bar{\mathcal{B}}^{k}(t) + \bar{\eta}^{k}\mathrm{T}^{k}(t) - M^{k}\ddot{U}^{k}(t) - K^{k}U^{k}(t) \\
M^{u} &= M^{22} - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}M^{12} \\
M^{k} &= M^{21} - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}M^{11} \\
K^{u} &= K^{22} - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}K^{12} \\
K^{k} &= K^{21} - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}K^{11} \\
\bar{\eta}^{k} &= \bar{\eta}^{21} - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}\eta^{11} \\
\bar{\mathcal{B}}^{k}(t) &= \mathcal{B}^{2}(t) - \bar{\eta}^{22}(\bar{\eta}^{12})^{-1}\mathcal{B}^{1}(t)
\end{aligned}$$

We now split the system (20) into elastic and electric parts as follows:

$$\begin{bmatrix} M^{u}_{uu} & 0 \\ M^{u}_{\varphi u} & 0 \end{bmatrix}\begin{bmatrix} \ddot{u}^{u}(t) \\ \ddot{\varphi}^{u}(t) \end{bmatrix} + \begin{bmatrix} K^{u}_{uu} & K^{u}_{u\varphi} \\ K^{u}_{\varphi u} & K^{u}_{\varphi\varphi} \end{bmatrix}\begin{bmatrix} u^{u}(t) \\ \varphi^{u}(t) \end{bmatrix} = \begin{bmatrix} Q^{k}_{u}(t) \\ Q^{k}_{\varphi}(t) \end{bmatrix} \tag{21}$$

The unknown electric potential $\varphi^{u}$ can be obtained from the second row of Eq. (21) as

$$\varphi^{u}(t) = \left(K^{u}_{\varphi\varphi}\right)^{-1}\left[Q^{k}_{\varphi}(t) - M^{u}_{\varphi u}\ddot{u}^{u}(t) - K^{u}_{\varphi u}u^{u}(t)\right] \tag{22}$$

With the aid of Eq. (22) into the first row of Eq. (21) we obtain

$$\bar{M}^{u}\ddot{u}^{u}(t) + \bar{K}^{u}u^{u}(t) = \bar{Q}^{k}(t) \tag{23}$$

where

$$\bar{Q}^{k}(t) = Q^{k}_{u}(t) - K^{u}_{u\varphi}\left(K^{u}_{\varphi\varphi}\right)^{-1}Q^{k}_{\varphi}(t)$$

$$\bar{M}^{u} = M^{u}_{uu} - K^{u}_{u\varphi}\left(K^{u}_{\varphi\varphi}\right)^{-1}M^{u}_{\varphi u}$$

$$\bar{K}^{u} = K^{u}_{uu} - K^{u}_{u\varphi}\left(K^{u}_{\varphi\varphi}\right)^{-1}K^{u}_{\varphi u}$$

We can write Eq. (23) at time step $n+1$

$$\bar{M}^u \ddot{u}^u_{\mathrm{n+1}} + \bar{K}^u u^u_{\mathrm{n+1}} = \bar{Q}^{\mathrm{k}}_{\mathrm{n+1}} \tag{24}$$

where

$$\bar{Q}^{\mathrm{k}}_{\mathrm{n+1}} = \bar{\mathcal{B}}^{\mathrm{k}}_{\mathrm{n+1}} + \eta^{\mathrm{k}} \mathrm{T}^{\mathrm{k}}_{\mathrm{n+1}} - M^k \ddot{u}^{\mathrm{k}}_{\mathrm{n+1}} - K^k u^{\mathrm{k}}_{\mathrm{n+1}} \tag{25}$$

The displacements $u_{n+1}$ and velocities $\dot{u}_{\mathrm{n+1}}$ used in this algorithm are approximated at time step n+1 as follows:

$$\dot{u}_{\mathrm{n+1}} \approx \dot{u}_{\mathrm{n}} + [(1-\delta)\ddot{u}_{\mathrm{n}} + \delta\ddot{u}_{\mathrm{n+1}}]\Delta\mathrm{t} \tag{26}$$

$$u_{n+1} \approx u_n + \dot{u}_{\mathrm{n}}\Delta\mathrm{t} + \left[\left(\frac{1}{2} - \alpha\right)\ddot{u}_{\mathrm{n}} + \alpha\ddot{u}_{\mathrm{n+1}}\right]\Delta\mathrm{t}^2 \tag{27}$$

The acceleration at time step n+1 may be expressed from Equation (27) as:

$$\ddot{u}_{\mathrm{n+1}} \approx \frac{1}{\alpha\Delta\mathrm{t}^2}(u_{n+1} - u_n) - \frac{1}{\alpha\Delta\mathrm{t}}\dot{u}_{\mathrm{n}} - \left(\frac{1}{2\alpha} - 1\right)\ddot{u}_{\mathrm{n}} \tag{28}$$

Upon substitution of (28) into (24) we obtain the following algebraic system

$$\mathbb{R} u^{\mathrm{u}}_{\mathrm{n+1}} = \mathcal{M}_{\mathrm{n+1}} \tag{29}$$

where the stiffness matrix $\mathbb{R}$ and effective load vector $\mathcal{M}_{\mathrm{n+1}}$ are given by

$$\mathbb{R} = \frac{1}{\alpha\Delta\mathrm{t}^2}\bar{M}^u + \bar{K}^u \tag{30}$$

$$\mathcal{M}_{\mathrm{n+1}} = \bar{Q}^{\mathrm{k}}_{\mathrm{n+1}} + \bar{M}^u\left[\frac{1}{\alpha\Delta\mathrm{t}^2}u^u_n + \frac{1}{\alpha\Delta\mathrm{t}}\dot{u}^u_n + \left(\frac{1}{2\alpha} - 1\right)\ddot{u}^u_n\right] \tag{31}$$

Once we have solved (29) for the unknown displacements at time step n+1, we can compute the accelerations and velocities from equations (28) and (26) respectively. Finally, the electric potential $\varphi^{\mathrm{u}}(t)$ can be obtained from (22) and the unknown generalized tractions $\mathrm{T}^u(\mathrm{t})$ can be determined using equation (19).

## 4 NUMERICAL RESULTS AND DISCUSSION

With the view of illustrating the numerical results, the material chosen for the plate is the piezoelectric ceramic Lead Zirconate Titanate (PZT), and the physical data for which is given as follows:

The elasticity tensor $C_{fghi}$ , piezoelectric tensor $e$ and relative permittivity $\epsilon^{\mathrm{rel}}$

$$C_{fghi} = \begin{pmatrix} 107.6 & 63.10 & 63.90 & 0.000 & 0.000 & 0.000 \\ 63.10 & 107.6 & 63.90 & 0.000 & 0.000 & 0.000 \\ 63.90 & 63.90 & 100.4 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 19.60 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 19.60 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 22.20 \end{pmatrix}$$

$$e = \begin{pmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 12.0 & 0.00 \\ 0.00 & 0.00 & 0.00 & 12.0 & 0.00 & 0.00 \\ -9.6 & -9.6 & 15.1 & 0.00 & 0.00 & 0.00 \end{pmatrix}$$

$$\epsilon^{\mathrm{rel}} = \begin{pmatrix} 1936 & 0.00 & 0.00 \\ 0.00 & 1936 & 0.00 \\ 0.00 & 0.00 & 2109 \end{pmatrix}$$

the results are plotted in Figs. 2–4 to show the validity of the DRBEM. These results obtained with the DRBEM have been compared graphically with those obtained using the Meshless Local Petrov–Galerkin (MLPG) method of Sladek et al. [47] are shown graphically in the same figures to confirm the validity of the proposed method. It can be seen from these figures that the DRBEM results are in excellent agreement with the results obtained by MLPG. The effects of the number of elements used were also examined. It was found that a further increase of boundary elements in the DRBEM led to improved numerical results (see Figures 2, 3 and 4).

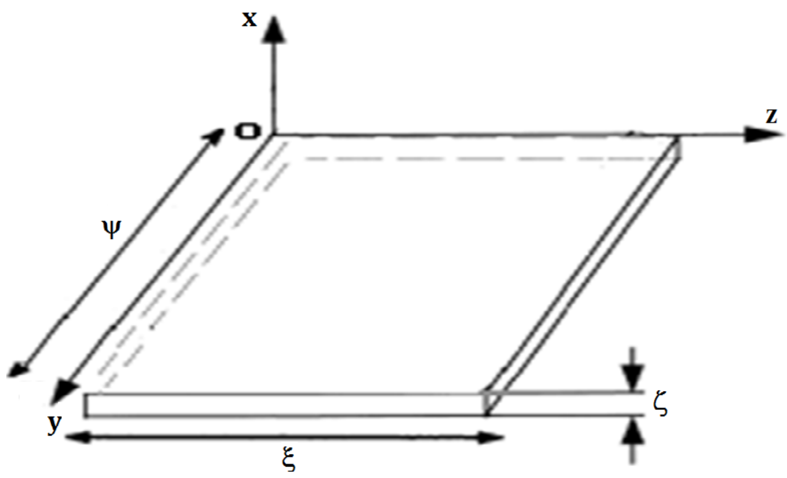


**Fig. 1. The coordinate system of the plate.**

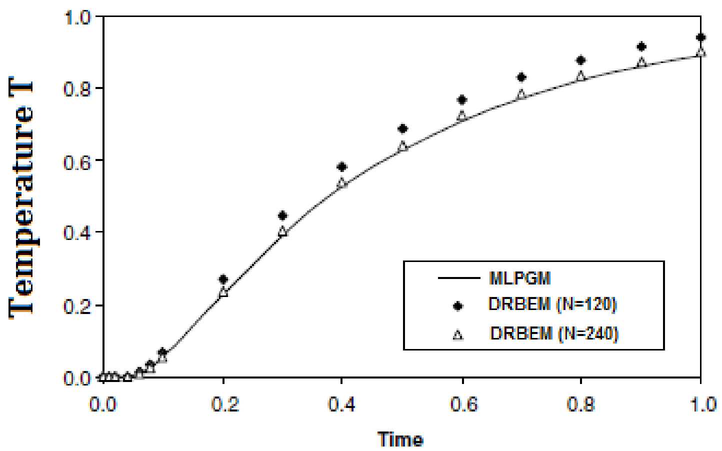


**Fig. 2. Variation of the temperature T with time t for MLPGM and DRBEM**

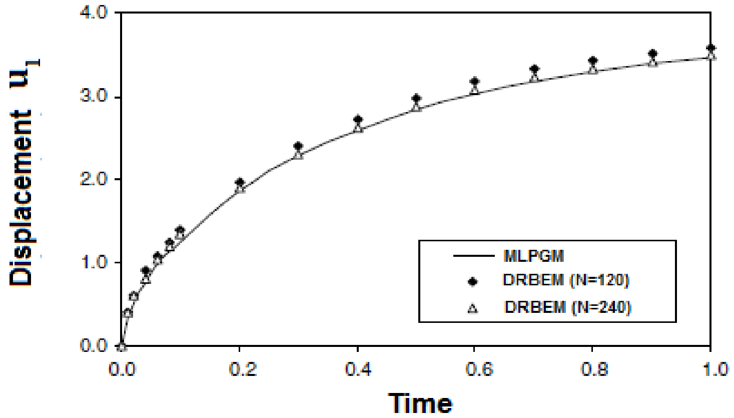


**Fig. 3. Variation of the displacement $u_1$ with time t for MLPGM and DRBEM**

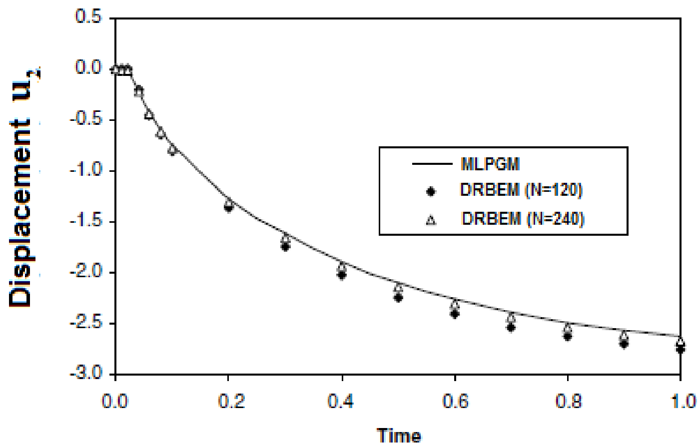


**Fig. 4. Variation of the displacement $u_2$ with time t for MLPGM and DRBEM**

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